MATH2050B 1920 Midterm

TA's solutions[1] to selected problems

Q1. State (without proof) the following results:

(i) Characterization Theorem for intervals

(ii) Nested Interval Theorem

(iii) Bolzano-Weierstrass Theorem

Solution.

(i) A subset $I \subset \mathbb{R}$ is an interval iff for any $a, b \in I$, whenever $a < x < b$ then $x \in I$.

(ii) Let $(I_n)_{n=1}^{\infty}$ be a sequence of decreasing closed and bounded intervals, i.e. $\forall n$, $I_n = [a_n, b_n]$, $a_n, b_n \in \mathbb{R}$ and $I_1 \supset I_2 \supset \ldots$. Then $\cap_{n=1}^{\infty} I_n \neq \varnothing$.

Moreover if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, then $\cap_{n=1}^{\infty} I_n = \{\xi\}$ (contains one and only one element)

(iii) Any bounded sequence of real numbers has a convergent subsequence.

Q2. Using Q1(iii) or other methods, prove the Cauchy criterion (if and only if) result for sequences.

Solution. **Cauchy criterion.** A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is convergent iff it is Cauchy.

First, we prove that if $(x_n)_{n=1}^{\infty}$ is convergent then it is Cauchy. Let $\epsilon > 0$. Suppose $\lim_{n\to\infty} x_n = L$. Then there is N s.t. $|x_n - L| < \frac{\epsilon}{2}$ for all $n > N$. Then for all $m, n > N$, $|x_m - x_n| \leq |x_m - L| + |L - x_n| < \epsilon$.

Second, we prove that if $(x_n)_{n=1}^{\infty}$ is Cauchy then it is convergent. To use BW Theorem we need to show that $(x_n)_{n=1}^{\infty}$ is bounded.

Consider $\epsilon_0 = 1$. By Cauchy condition there is N s.t. $|x_n - x_m| < 1 = \epsilon_0$ for all $m, n > N$. In particualr, $|x_n - x_{N+1}| < 1$ for all $n > N$. Thus $|x_n| < |x_{N+1}| + 1$ for all $n > N$. Therefore $(x_n)_{n=1}^{\infty}$ is bounded by $\max\{|x_1|, |x_2|, \ldots, |x_N|, |x_{N+1}| + 1\}$.

By BW Theorem, $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, say $\lim_{k\to\infty} x_{n_k} = L$. It remains to show $\lim_{n\to\infty} x_n = L$.

Let $\epsilon > 0$.

- By Cauchy condition, there is $M_1 \in \mathbb{N}$ s.t. $|x_m - x_n| < \frac{\epsilon}{2}$ for all $n, m \geq M_1$.
- By convergence of $(x_{n_k})_{k=1}^{\infty}$, there is $M_2 \in \mathbb{N}$ s.t. $|x_{n_k} - L| < \frac{\epsilon}{2}$ for all $k \geq M_2$.

[1]please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Let $M = \max(M_1, M_2)$. For any $n > M$,

$$|x_n - L| \le |x_n - x_{n_{M_2}}| + |x_{n_{M_2}} - L| < \epsilon.$$

Q3. For a real-valued function f on a set of real numbers, give the definition and its negation for each of the following:

(i) f is continuous at a point (say u) in A

(ii) f uniformly continuous on A

Solution.

(i) (Definition) f is continuous at $u \in A$ iff for any $\epsilon > 0$, there is $\delta > 0$ s.t. for all $x \in V_\delta(u) \cap A$, $|f(x) - f(u)| < \epsilon$.

(Negation) f is not continuous at $u \in A$ iff there is $\epsilon > 0$ s.t. for any $\delta > 0$, there is $x \in V_\delta(u) \cap A$ with $|f(x) - f(u)| \ge \epsilon$.

(ii) (Definition) f is uniformly continuous on A iff for any $\epsilon > 0$, there is $\delta > 0$ s.t. for all $x, y \in A$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

(Negation) f is not uniformly continuous on A iff there is $\epsilon > 0$ s.t. for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ and $|f(x) - f(y)| \ge \epsilon$.

Q4. Let $f(x) = x^2$, $\forall x \in \mathbb{R}$. Show that f is continuous at each point of $\mathbb{R}$ and that is is not uniformly continuous.

Solution. The identity function $g(x) = x$ is continuous on $\mathbb{R}$, so product of continuous functions $f = g^2$ is continuous on $\mathbb{R}$.

To show that f is not uniformly continuous on $\mathbb{R}$, we use the negation stated as in **Q3** (ii).

Consider $\epsilon = 1$. For any $\delta > 0$, we want to find $x, y \in \mathbb{R}$, $|x - y| < \delta$ and $|f(x) - f(y)| \ge 1$.

Note that there exists $n \in \mathbb{N}$ s.t. $\delta n > 1$. Now, the two real numbers n, $n + \frac{\delta}{2}$ is of distance $< \delta$, and

$$|f(n) - f(n + \frac{\delta}{2})| = \delta n + \frac{\delta^2}{4} > 1$$

Hence f is not uniformly continuous on $\mathbb{R}$.

Q5. In ϵ-δ terminology, show:

(i) $\lim_{x \to 4} \frac{x^2+1}{x-3} = 17$

(ii) $\lim_{x \to 2+} \frac{x}{x-2} = +\infty$, $(x > 2)$

(iii) If $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous then $\frac{1}{f}$ is continuous.

(iv) If $f, g : \mathbb{R} \to \mathbb{R}$ continuous then $f \circ g$ is continuous on $\mathbb{R}$, where $(f \circ g)(t) = f(g(t))$, $\forall t \in \mathbb{R}$

Solution. (i) : Note $|\frac{x^2+1}{x-3} - 17| = |\frac{(x-4)(x-13)}{x-3}|$. If $0 < |x-4| < \frac{1}{2}$, then $|x-3| > \frac{1}{2}$ and $|x-13| < \frac{17}{2}$.

Let $\epsilon > 0$. Take $\delta = \min(\frac{1}{2}, \frac{\epsilon}{17})$. For any $0 < |x-4| < \delta$, we have

$$|\frac{x^2+1}{x-3} - 17| = |\frac{(x-4)(x-13)}{x-3}| < \epsilon.$$

(ii) : Let $M \in \mathbb{R}$. We need to show that there is $\delta > 0$ s.t. for all x with $0 < x-2 < \delta$, $\frac{x}{x-2} > M$. WLOG we may assume $M > 1$. Then set $\delta = \frac{2}{M-1}$. For any x with $0 < x-2 < \delta = \frac{2}{M-1}$, we have

$$x < 2 + \frac{2}{M-1} = \frac{2M}{M-1}$$

Thus $Mx - x < 2M \Rightarrow M < \frac{x}{x-2}$.

(iii): Let $x_0 \in \mathbb{R}$ be fixed, we need to show $\frac{1}{f}$ is continuous at x_0. Because $f(x_0) \neq 0$. By continuity, there is $\delta' > 0$ s.t. for all x with $|x - x_0| < \delta'$, $|f(x)| > \frac{|f(x_0)|}{2} > 0$.

Let $\epsilon > 0$. By continuity of f there is $\delta'' > 0$ s.t. for all x with $|x - x_0| < \delta''$,

$$|f(x) - f(x_0)| < \epsilon\frac{|f(x_0)|^2}{2}$$

Take $\delta = \min(\delta', \delta'')$. For any x with $|x - x_0| < \delta$,

$$|\frac{1}{f(x)} - \frac{1}{f(x_0)}| = \frac{|f(x) - f(x_0)|}{|f(x)f(x_0)|} < \frac{2}{|f(x_0)|^2}|f(x) - f(x_0)| < \epsilon.$$

(iv): Please refer to Theorem **5.2.6** and **5.2.7** of Bartle and Sherbert's Introduction to Real Analysis.